# CVXPY: A Rewriting System for Convex Optimization

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# Convex optimization

# Optimization problem

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\
& g_i(x) = 0, \quad i = 1, \dots, p
\end{array}
$$

- $x \in \mathbf{R}^n$ is (vector) variable to be chosen
- $f_0$ is the *objective function*, to be minimized
- $f_1, \dots, f_m$ are the *inequality constraint functions*
- $g_1, \dots, g_p$ are the *equality constraint functions*

**Goal:** find a value for $x$ that minimizes $f_0(x)$, while satisfying constraints

# Convex optimization problem

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& Ax = 0
\end{array}
$$

- equality constraint functions are affine
- $f_0, \ldots, f_m$ are **convex**: for $\theta \in [0, 1]$,

$$f_i(\theta x + (1-\theta)y) \leq \theta f_i(x) + (1-\theta) f_i(y)$$

*i.e.*, $f_i$ curve upward

# Why convex optimization?

- solution algorithms that work well, in theory and practice
- **many applications** in
  - machine learning, statistics
  - control
  - signal, image processing
  - networking
  - engineering design
  - finance

...and many more

# How do you solve a convex optimization problem?

use someone else's ('standard') solver

- your problem *must* be written in a standard form
- analogous to writing machine code

write your own (custom) solver

- lots of work, but can take advantage of special structure

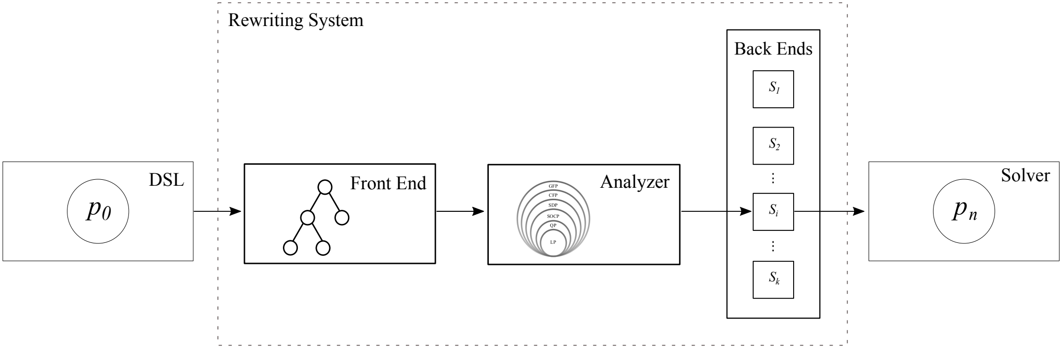
**this talk: use a domain-specific language**

- transforms user-friendly format into solver-friendly standard form
- extends reach of problems solvable by standard solvers

# Domain-specific languages

# Domain-specific languages (DSLs)

- DSLs make it easy to specify and solve convex problems
- Grammar and semantics based on a rule from convex analysis [GBY06]
- Examples: CVXPY, CVXR, Convex.jl, CVX



# Example

CVXPY is a Python-embedded DSL [DB16; AVD+18]

```python
import cvxpy as cp

x = cp.Variable()
y = cp.Variable()

objective = cp.Minimize(cp.maximum(x + y + 2, -x - y))
constraints = [x <= 0, y == -0.5]

problem = cp.Problem(objective, constraints)
assert problem.is_dcp()
optimal_value = problem.solve()
```

# Example

- First, CVXPY **analyzes** the problem and checks that it's valid (convex)
- Next, CVXPY **reduces** the problem into a low-level form

  *e.g.*, `problem` is equivalent to a linear program, with form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \leq h \\ & Ax = b, \end{array}$$

$$G = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad h = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \quad b = -0.5$$

- Finally, CVXPY solves the problem via a numerical solver

# Analysis

# Grammar

- CVXPY's grammar consists of atomic functions (**atoms**) and a rule for combining them
- atoms have known curvature (convex, concave, affine) and monotonicity (increasing, decreasing) (`log`, `exp`, `square`, `sum`, ...)
- rule guarantees that compositions of atoms have known curvature
- grammar is called **disciplined convex programming**

**Composition rule**: $h(f_1(x), \ldots, f_k(x))$ is convex when $h$ is convex and for each $i$

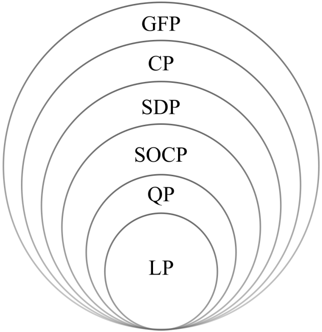
- $h$ is increasing in argument $i$, and $f_i$ is convex, or
- $h$ is decreasing in argument $i$, and $f_i$ is concave, or
- $f_i$ is affine

# Analysis

- A problem object is represented as a collection of expression trees
- Nodes are atoms; leaves are variables and numeric constants
- Each tree represents a composition of atoms
- CVXPY checks whether a problem is DCP by recursively checking the composition rule for each tree

# Analysis

- Convex optimization problems can be organized into a hierarchy of classes
- Different solvers support different classes
- Each supported class has its own grammar (typically a subset of DCP)
- By default, CVXPY targets the most specific class possible



# Other grammars

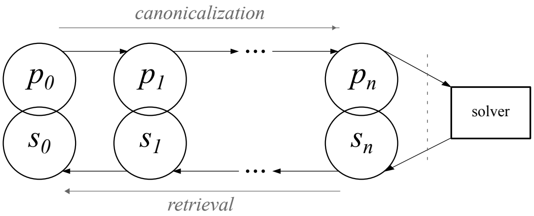
- CVXPY is easily extended to support grammars other than DCP
- *e.g.*, DGP [ADB19] for geometric programs, DQCP [AB20] for quasiconvex programs, ...
- Grammar-checking implementation uncoupled from atom implementation
- Decision to add new grammar based on various factors
    - usefulness
    - implementation ease
    - maintainability
    - taste / aesthetics

# Reductions

# Reductions

- CVXPY transforms the original problem via a sequence of **reductions**
- A reduction converts a problem into a different but equivalent problem
- Each reduction has three methods:

  `accepts`, which defines the class of problems the reduction can accept
  `reduce`, which takes a problem and reduces it to another
  `retrieve`, which retrieves a solution to the original problem from a solution to the emitted problem



# Reductions

Some examples:

- `FlipObjective`
- `Complex2Real`
- `Dcp2Cone`
- `Dgp2Dcp`
- `Dqcp2Dcp`
- `EliminatePwl`
- `Qp2SymbolicQp`
- `MatrixStuffing`, `ConeMatrixStuffing`, `QpMatrixStuffing`
- change of variables, presolves, and more ...

# Reductions

A simple example: the `FlipObjective` reduction.

Accepts:

$$\begin{array}{ll} \text{maximize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = 0 \end{array}$$

Reduces to:

$$\begin{array}{ll} \text{minimize} & -f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = 0 \end{array}$$

Retrieval: a no-op (solutions are identitical)

# Standard form

- A chain of reductions ends with a targeted standard form
- The modern standard form is the *cone program*

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b, \quad x \in \mathcal{K}
\end{array}
$$

  where $\mathcal{K}$ is a Cartesian product of convex cones
- Special cases include linear programs, semidefinite programs
- There are several solvers for cone programs (SCS, ECOS, MOSEK, ...)

# Other standard forms

Reduction system makes it easy to add other problem classes

- quadratic programs:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x \\ \text{subject to} & l \leq Ax \leq u \end{array}$$

- linearly constrained least squares:

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & Fx = g \end{array}$$

- nonlinear programs:

$$\text{minimize} \quad f(x)$$

where $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable

# Reduction chains

Reductions are chained together to target a standard form

- `Qp2SybmolicQp` $\to$ `QpMatrixStuffing` $\to$ `OSQP`
- `Complex2Real` $\to$ `Dcp2Cone` $\to$ `ConeMatrixStuffing` $\to$ `ECOS`
- `Dgp2Dcp` $\to$ `Dcp2Cone` $\to$ `ConeMatrixStuffing` $\to$ `SCS`

CVXPY builds these reduction chains automatically, behind-the-scenes

# Recent extensions

# Parametrized programs

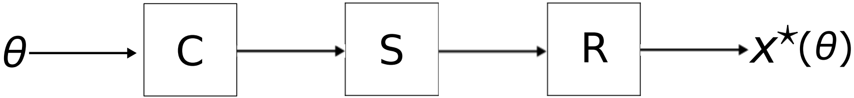
$$\begin{array}{ll} \text{minimize} & f_0(x;\theta) \\ \text{subject to} & f_i(x;\theta) \leq 0, \quad i = 1,\ldots,p \\ & A(\theta)x = b(\theta) \end{array}$$

- Objective function and constraints often depend on some numerical parameters $\theta$
- With some mild assumptions, the mapping from $\theta$ to problem data of the final reduced-to problem is *affine*
- We can represent CVXPY's rewriting by multiplication with a sparse matrix
- This fact enables two new features: efficiently differentiating through convex optimization problems, and code generation

# Differentiating through CVXPY

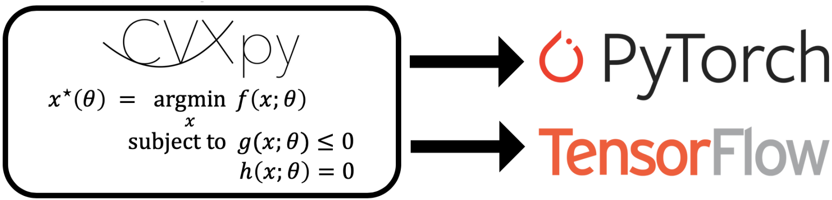
Solution map of a parametrized CVXPY problem: $x^\star(\theta) = (R \circ S \circ C)(\theta)$

- Problem is *canonicalized* (C) to a standard form
- The canonicalized problem is *solved* (S)
- A solution for the original problem is *retrieved* (R)
- We can efficiently differentiate through $C$, $S$, and $R$ [AAB+19]



# Exporting to PyTorch and TensorFlow

`cvxpylayers`: an open-source library for exporting CVXPY problems to PyTorch and TensorFlow




`https://github.com/cvxgrp/cvxpylayers`

# Tuning a Markowitz policy

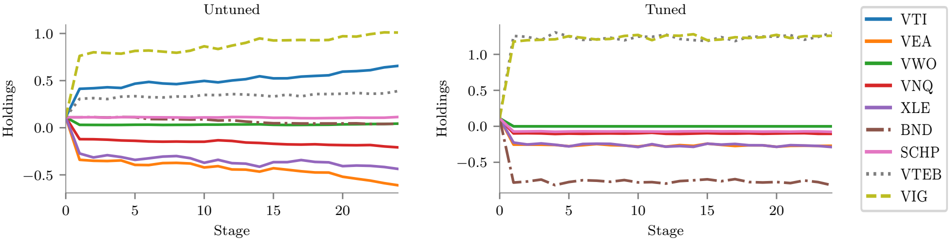



Figure: left: untuned; right: policy with tuned constraints, mean and covariance

# Tracking a vehicle trajectory

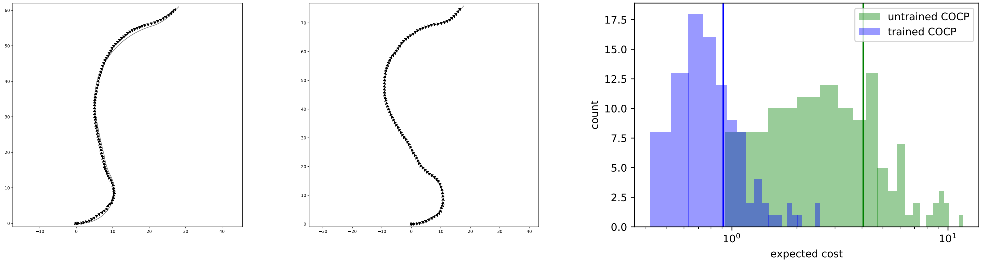



Figure: left: untrained path; middle: trained path: right: expected cost histogram.

# Code generation

- Extracts sparse matrices representing the rewriting
- Generates C code that takes parameters, calls solver, and returns solution
- Lets you deploy solvers in real-time applications
- Coming soon . . .

# Summary

CVXPY is a modular rewriting system for convex optimization that makes convex optimization more accessible to researchers and engineers alike by abstracting away low-level numerical solvers.

- Simple grammar lets users specify problems that are verifiably convex
- Analysis phase matches a high-level problem with a low-level problem class
- Reduction system makes it easy to add new grammars and solvers
- Rewriting is amenable to optimizations when parameters are used

https://cvxpy.org

# References


[AAB+19] A. Agrawal, B. Amos, S. Barratt, S. Boyd, S. Diamond, and Z. Kolter. Differentiable convex optimization layers. In *Advances in Neural Information Processing Systems*. 2019.

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